4. V. V. Levdansky, V. G. Leitsina, O. G. Martynenko, N. V. Pavlyukevich, and R. I. Soloukhin, "Radiative heat transfer in a model porous body," Proc. VIIth Int. Heat Transfer Conference, Vol. 2, München (1982), pp. 523-527.

STABLE CONSERVATIVE DIFFERENCE SCHEMES FOR THE QUASILINEAR PARABOLIC HEAT-CONDUCTION EQUATION

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An efficient algorithm is developed for solving the quasilinear heat-conduction equation using asymmetric difference schemes satisfying the discrete analog of the conservation law.

The optimum regimes of plasma-mechanical treatment (PMT) [1, 2] depend essentially on the temperature field in the surface layer [3]. A more careful examination of processes of interaction of high-intensity heat fluxes with solids leads to the necessity of allowing for the temperature dependence of the thermophysical properties of the material being treated. In order to eliminate structural phase transitions in brittle metals during PMT [4], as well as to prevent the process of development of destructive temperature stresses, it is necessary to provide conditions for the heating of only that part of the volume of the component which is subject to removal [1]. It is proposed to calculate the temperature field in an isolated volume of a half-space by an approximate numerical solution of the quasilinear heat-conduction equation [5], with boundary conditions of the first kind, using explicit, absolutely stable, asymmetric difference schemes (ADS), satisfying a certain discrete analog of the law of conservation of energy, applying averaging by the arithmetic-mean method [6]. The initial ADS are obtained from the heat-conduction equation using the integrointerpolation method [5] with subsequent splitting [7].

It is well known that the numerical solution of problems of mathematical physics imposes especially rigorous demands on both the memory volume and the operating speed of computers [8]. A promising method of overcoming the difficulties arising in the solution of problems of mathematical physics is the use of parallel multiprocessor computer systems [9]. The ADS method has an algorithmic structure not requiring a preliminary procedure of conversion of a sequential algorithm into a parallel one [10] for the programmed execution on multiprocessor computers. The reduction of computer time in the use of multiprocessor computers plays an especially important role in the solution of nonlinear multidimensional problems of mathematical physics.

Let us consider the quasilinear heat-conduction equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x} k(T) \frac{\partial T}{\partial x}, \tag{1}$$

where $T \geqslant 0$ is the temperature, while the dependence $k(T) \geqslant 0$ of the coefficient of thermal conductivity is assumed to be given. For the unique solvability of the problem it is necessary to assign the boundary conditions

$$T(0, t) = f_1(t), \quad T(l, t) = f_2(t) \tag{2}$$

and initial conditions

$$T(x, 0) = \varphi(x). \tag{3}$$

Introducing the heat-flux function

$$W = k(T) \frac{\partial T}{\partial x}, \tag{4}$$

we rewrite Eq. (1) in the flux form

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$$\frac{\partial T}{\partial t} = \frac{\partial W}{\partial x}. \tag{5}$$

We approximate Eqs. (4) and (5) on a grid with nodes $x_i = ih$, $i = 1, 2, N - 1$, $h = l/N$, $t_k = k\tau$, $k = 0, 1, \ldots$. We apply the integrointerpolation method [5] to Eq. (5). For this we integrate (5) first over time in the band $t_k < t < t_{k+1}$ and then over the spatial coordinate in the band $x_{i-1/2} < x < {}_{i+1/2}$. Applying the simplest quadrature formulas and carrying out averaging on the left side of the equation, we arrive at an expression of the type

$$\hat{T} - T = \frac{\tau}{2h}(\hat{W}_{1/2} - \hat{W}_{-1/2} + W_{1/2} - W_{-1/2}), \tag{6}$$

where the following notation without indices is used for convenience in writing: $\hat{T} \equiv T_i^{k+1}$, $T \equiv T_i^k$, $\hat{W}_{1/2} \equiv W_{i+1/2}^{k+1}$, $W_{-1/2} \equiv W_{i-1/2}^k$, etc. Equation (6) is a uniform and conservative difference scheme for the quasilinear heat-conduction equation (5) with the condition that Eq. (4) is also approximated on a grid using the integrointerpolation method [5]. Therefore, we approximate the heat fluxes as

$$\hat{W}_{1/2} = \hat{a}_{1/2}\frac{\hat{T}_+ - \hat{T}}{h}, \quad W_{-1/2} = a_{-1/2}\frac{T - T_-}{h} \tag{7}$$

etc., where $\hat{a}_{1/2} = a(\hat{T}_+, \hat{T})$, $a_{-1/2} = a(T, T_-)$, etc.; a is a certain functional approximating the coefficient of thermal conductivity on a grid with an error $O(h^2)$ [5]. As one would expect, calculations showed that the best results are obtained by using equations of the type

$$\hat{a}_{1/2} = \frac{\hat{k}_+ + \hat{k}}{2}. \tag{8}$$

We apply the method of fractional steps [11] to the difference scheme (6). Splitting Eq. (6) yields two equations:

$$\hat{T} - T = \frac{\tau}{h}(\hat{W}_{1/2} - W_{-1/2}), \tag{9}$$

$$\hat{T} - T = \frac{\tau}{h}(W_{1/2} - \hat{W}_{-1/2}). \tag{10}$$

Substituting equations of the type (7), (8) into (9) and (10), we obtain two explicit asymmetric difference schemes:

$$\hat{\vec{T}} = \vec{A}T_+ + \vec{B}T + \vec{C}\hat{T}_-, \tag{11}$$

$$\hat{\overleftarrow{T}} = \overleftarrow{A}\hat{T}_+ + \overleftarrow{B}T + \overleftarrow{C}T_-. \tag{12}$$

The arrows indicate the direction of calculation: A calculation by Eq. (11) is carried out from left to right while one by Eq. (12) is carried out from right to left. The end result at the top time layer is found by the arithmetic-mean method [6]:

$$\hat{T} = (\hat{\vec{T}} + \hat{\overleftarrow{T}})/2. \tag{13}$$

The coefficients A, B, and C in schemes (11) and (12) are calculated from the formulas

$$\begin{gathered}\vec{A} = \vec{\mu}\gamma a_{1/2}, \quad \vec{B} = \vec{\mu}(1 - \gamma a_{1/2}), \quad \vec{C} = \vec{\mu}\gamma\hat{a}_{-1/2},\\ \overleftarrow{A} = \overleftarrow{\mu}\gamma\hat{a}_{1/2}, \quad \overleftarrow{B} = \overleftarrow{\mu}(1 - \gamma a_{-1/2}), \quad \overleftarrow{C} = \overleftarrow{\mu}\gamma a_{-1/2},\\ \vec{\mu} = 1/(1 + \gamma\hat{a}_{-1/2}), \quad \overleftarrow{\mu} = 1/(1 + \gamma\hat{a}_{1/2}).\end{gathered} \tag{14}$$

Comment. The presence of the functionals $\hat{a}_{1/2}$ and $\hat{a}_{-1/2}$, approximating the values of the nonlinear coefficient of thermal conductivity in the top time layer, on the right sides of Eqs. (14) leads to the necessity of introducing an iteration process for the calculation of Eqs. (14). The initial approximation in the top time layer at all the spatial nodes except the two boundary nodes is found as follows. Using a Taylor series expansion, it is easy to show that

$$\hat{T}_- = (\hat{T}_{-2} + \hat{T})/2 + O(h^2), \quad \hat{T}_+ = (\hat{T} + \hat{T}_{+2})/2 + O(h^2). \tag{15}$$

In fact, for the first equation in (15), for example, we have

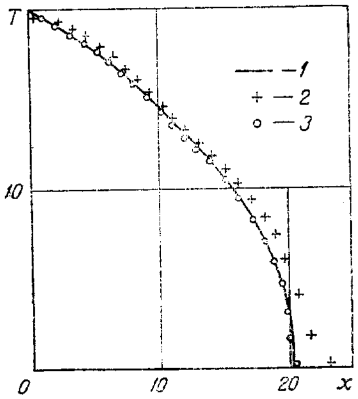

Fig. 1. Results of the comparison: 1) analytic solution; 2) nonconservative solution; 3) conservative solution. x, m; T, deg.

$$\hat{T}_{-2} = \hat{T}_{-} - h\frac{\partial \hat{T}_{-}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \hat{T}_{-}}{\partial x^2} + O(h^3),$$

$$\hat{T} = \hat{T}_{-} + h\frac{\partial \hat{T}_{-}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \hat{T}_{-}}{\partial x^2} + O(h^3).$$

Combining the two latter expressions, we obtain the first of the equations (15). The second equation is demonstrated similarly. From (15) it follows that

$$\vec{\hat{T}} = 2\hat{T}_{-} - \hat{T}_{-2} + O(h^2), \quad \overleftarrow{\hat{T}} = 2\hat{T}_{+} - \hat{T}_{+2} + O(h^2).$$

And these expressions represent the initial approximations being sought. At the boundary nodes we assume, for example, that

$$\hat{T}_2 = (\hat{T}_1 + T_2)/2, \quad \hat{T}_N = (T_N + \hat{T}_{N+1})/2.$$

The presence of the iteration process can lead to violation of the heat balance even when using conservative difference schemes [12]. To eliminate this defect it is proposed to use a discrete analog of the law of conservation of energy at each time layer, and for the difference scheme under consideration it can be represented in the form

$$\sum_{i=1}^{N^*} T_i^{k+1} = \sum_{i=1}^{N} T_i^1 + \frac{\tau}{h}\sum_{m=1}^{k+1}(W_{N+0.5}^m - W_{1.5}^m). \tag{16}$$

The first sum on the right of the equals sign is found once in the calculation of the initial conditions. The remaining sum on the right side of the equation is calculated at each time layer after the execution of the algorithm (11)-(13). Then, simultaneously with the transition to the next time layer, summation is carried out on the left side of Eq. (16), until approximate equality is reached for a certain $i = N^*$. The remaining values of T_i^{k+1} are equated to zero for all $i > N^*$. The violation of heat balance is thereby eliminated. A computational experiment, a description of which is given below, confirmed the efficiency of the application of the discrete analog of the law of conservation of energy to restore the conservativity of scheme (6) violated in the use of the iteration process.

The computational experiment was carried out with an equation of the type (1) with a coefficient of thermal conductivity $k = k_0T^s$, $k_0 = 0.5$, $s = 2$ [13]. The boundary conditions were chosen in the form

$$T(0,\ t) = 10\sqrt{t}, \quad T(l,\ t) = 0, \tag{17}$$

while the initial conditions were determined, as in [14], from an analytic solution representing a thermal wave traveling along a zero temperature front with a velocity V,

$$T(x,\ t) = \begin{cases} \left[\dfrac{sV}{k_0}(Vt - x_1 - x)\right]^{1/s} & \text{at } x \leqslant x_1 + Vt \\ 0 & \text{at } x > x_1 + Vt \end{cases} \tag{18}$$

at the time $t = 0.01$. The velocity is $V = 5$.

The results of the comparison with the test problem are presented in Fig. 1. The calculation was made up to the time $t = 4.01$ with steps of $\tau = 0.08$ and $h = 1$. The comparison was made with the analytic solution (18), as well as with a grid function obtained using the PKG1 subprogram from the packet of scientific subprograms developed by the Institute of Mathematics, Academy of Sciences of the Belorussian SSR [15]. The analytic solution is shown by the solid line in Fig. 1, the nonconservative grid function is denoted by the symbol +, and the conservative grid function by the symbol o. The PKG1 subprogram gives almost the same error as the ADS with restoration of conservativity. From all the foregoing, it follows that conservative ADS can be used to calculate thermal waves on actual grids with sufficient accuracy for practical needs.

Thus, a means of constructing conservative, explicit, absolutely stable ADS for the quasilinear heat-conduction equation using the integrointerpolation method with splitting is proposed in the paper. The efficiency of the use of the discrete analog of the law of conservation of energy to restore conservativity, violated due to the application of the iteration process, is illustrated on a numerical example. The absolute stability of the ADS can be shown using the operator-difference method proposed by Samarskii [5]. In working on the paper the authors were guided by an idea advanced by Kolesnikov [16, 17] on the promising nature of the application of methods of gas dynamics for purposes of modeling high-intensity thermophysical processes. Certain experimentally revealed limits on the relation between the time step and the spatial step, which was also noted by other authors [18], should be mentioned.

Analogous results can be obtained not only for Eq. (1) but also for other equations of mathematical physics (including multidimensional and nonlinear equations). For example, a stable ADS for the two-dimensional quasilinear heat-conduction equation

$$\frac{\partial T}{\partial t}=\frac{\partial WX}{\partial x}+\frac{\partial WY}{\partial y}, \tag{19}$$

where $WX = k(T)\partial T/\partial x$ and $WY = k(T)\partial T/\partial y$ are heat-flux functions, looks like this:

$$\begin{aligned}
\hat{T}_1&=\mu_1\{T+GX[\hat{A}_{1/2}\hat{T}_+-A_{-1/2}(T-T_-)]+GY[\hat{A}_{,1/2}\hat{T}_{,+}-A_{,-1/2}(T-T_{,-})]\},\\
\hat{T}_2&=\mu_2\{T+GX[A_{1/2}(T_+-T)+\hat{A}_{-1/2}\hat{T}_-]+GY[A_{,1/2}(T_{,+}-T)+\hat{A}_{,-1/2}\hat{T}_{,-}]\},\\
\hat{T}_3&=\mu_3\{T+GX[\hat{A}_{1/2}\hat{T}_+-A_{-1/2}(T-T_-)]+GY[A_{,1/2}(T_{,+}-T)+\hat{A}_{,-1/2}\hat{T}_{,-}]\},\\
\hat{T}_4&=\mu_4\{T+GX[A_{1/2}(T_+-T)+\hat{A}_{-1/2}\hat{T}_-]+GY[\hat{A}_{,1/2}\hat{T}_{,+}-A_{,-1/2}(T-T_{,-})]\}.
\end{aligned} \tag{20}$$

The end result at the top time layer is found using the expression

$$\hat{T}=(\hat{T}_1+\hat{T}_2+\hat{T}_3+\hat{T}_4)/4. \tag{21}$$

In this case the coefficients μ and A are calculated from the equations

$$\begin{gathered}
\mu_1=1/(1+GX\cdot\hat{A}_{1/2}+GY\cdot\hat{A}_{,1/2}),\quad \mu_2=1/(1+GX\cdot\hat{A}_{-1/2}+GY\cdot\hat{A}_{,-1/2}),\\
\mu_3=1/(1+GX\cdot\hat{A}_{1/2}+GY\cdot\hat{A}_{,-1/2}),\quad \mu_4=1/(1+GX\cdot\hat{A}_{-1/2}+GY\cdot\hat{A}_{,1/2}),\\
A_{1/2}=(k+k_+)/2,\quad A_{-1/2}=(k_-+k)/2,\quad A_{,1/2}=(k+k_{,+})/2,\\
\hat{A}_{1/2}=(\hat{k}+\hat{k}_+)/2,\quad \hat{A}_{-1/2}=(\hat{k}_-+\hat{k})/2,\quad \hat{A}_{,1/2}=(\hat{k}+\hat{k}_{,+})/2,\\
A_{,-1/2}=(k_{,-}+k)/2,\quad \hat{A}_{,-1/2}=(\hat{k}_{,-}+\hat{k})/2.
\end{gathered} \tag{22}$$

The comment made with regard to Eqs. (14) also applies in equal measure to Eqs. (22).

Boundary conditions of the first kind, which are not written out here, are understood in the algorithm (20)-(22) of the solution of the two-dimensional quasilinear heat-conduction equation using the ADS. Generalization of the proposed method of derivation of the ADS for the case of boundary conditions of the second or third kind can be done directly.

NOTATION

T, temperature; t, time; x, spatial coordinate; k(T), coefficient of thermal conductivity; $f_1(t)$, $f_2(t)$, φ(x), functions; l, length of the interval of integration; W, heat flux; h, τ, steps of the grid along the spatial and time coordinates; i, k, indices; a, functional approximating the value of the coefficient of thermal conductivity on the difference grid; $\gamma = \tau/h^2$, Courant number; μ, coefficient; N*, number of the node at which the equality (16) is (approximately) reached; k_0, coefficient; s, exponent; V, velocity of propagation of the thermal wave; x_1, origin; WX, WY, two-dimensional heat-flux functions; GX = τ/HX^2, GY = τ/HY^2, Courant numbers (along the coordinates); HX, HY, steps of the two-dimensional difference grid along the X and Y axes.

LITERATURE CITED

1. A. E. Pankov and V. S. Poluyanov, Plasma-Mechanical Treatment of Metals (Survey) [in Russian], Nauchno-Issled. Inst. Mash., Moscow (1981).
2. Practical Application of Plasma-Mechanical Treatment in Power Mechanical Engineering [in Russian], Nauchno-Issled. Inst. Energ. Inform. Énergomashinostr., Moscow (1980).
3. V. I. Gladkovskii, M. I. Sazonov, and N. I. Chopchits, "Change in the surface strength of metals during heating by a rapidly moving heat source," in: Abstracts of Papers of the 10th All-Union Conference on the Physics of the Strength and Plasticity of Metals and Alloys, July 21-23, 1983, Kuibyshev: Scientific Council on the Comprehensive Problem "Solid-State Physics," Academy of Sciences of the USSR, Ministry of Higher Education of the RSFSR [in Russian], Kuibyshev Polytekh. Inst. (1983), p. 358.
4. V. I. Gladkovskii and V. G. Karolinskii, "Application of asymmetric difference schemes for the solution of the Stefan problem of structural phase transitions," in: High-Speed Processes in Thermal and Mechanical Action on Metallic Materials, Summaries of Papers of Scientific—Technical Conference, Minsk, October 25-26, 1984 [in Russian], Belorus. Nauchno-Issled. Inst. Nauchno-Tekh. Inform. Tekh.-Ékon. Issled., Gosplana Belorus. SSR, Minsk (1984), pp. 93-94.
5. A. A. Samarskii, Theory of Difference Schemes [in Russian], Nauka, Moscow (1979).
6. V. K. Saul'ev, "Application of asymmetric difference approximations for solving the Burgers equation," Differents. Uravn., 18, No. 12, 2190-2195 (1982).
7. A. A. Samarskii, Introduction to Numerical Methods [in Russian], Nauka, Moscow (1982).
8. K. I. Babenko (ed.), Theoretical Principles and the Construction of Numerical Algorithms for Problems of Mathematical Physics [in Russian], Nauka, Moscow (1979).
9. B. A. Golovkin, Parallel Computing Systems [in Russian], Nauka, Moscow (1980).
10. I. V. Prangishvili, S. Ya. Vilenkin, and I. L. Medvedev, Parallel Computing Systems with a Comon Control [in Russian], Énergoatomizdat, Moscow (1983).
11. N. N. Yanenko, Method of Fractional Steps in the Solution of Multidimensional Problems of Mathematical Physics [in Russian], Nauka, Novosibirsk (1967).
12. M. Yu. Shashkov, "Violation of conservation laws in the solution of difference equations by iteration methods," Zh. Vychisl. Mat. Mat. Fiz., 22, No. 5, 1149-1156 (1982).
13. A. A. Samarskii and I. M. Sobol', "Numerical examples of the calculation of temperature waves," Zh. Vychisl. Mat. Mat. Fiz., 3, No. 4, 702-719 (1963).
14. V. N. Abrashin, "Stable difference schemes for quasilinear equations of mathematical physics," Differents. Uravn., 18, No. 11, 1967-1971 (1982).
15. Mathematical Software of the ES Computer Academy of Sciences of the Belorussian SSR, Institute of Mathematics [in Russian], Inst. Mat., Minsk (1981), Part 28.
16. P. M. Kolesnikov, "Simple and shock waves in a nonlinear, high-intensity, nonsteady process of heat and mass transfer," Inzh.-Fiz. Zh., 15, No. 3, 501-504 (1968).
17. P. M. Kolesnikov, Methods of Transfer Theory in Nonlinear Media [in Russian], Nauka i Tekhnika, Minsk (1981).
18. V. K. Saul'ev and A. A. Chernikov, "Solution by the finite-difference method of a regularized equation for shallow water," Differents. Uravn., 19, No. 10, 1818-1820 (1983).